Math 102

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Goals Today

- Global maxima and minima of a function over an interval
- Introduction to Optimization (Unconstrained and Constrained)
  - Determine the **objective function**: what quantity are we trying to optimize?
  - Determine the **variables**: what quantities determine the objective function?
  - Determine the **constraints**: what are the constraining relationships between the variables?

Optimization

- Largest possible volume of a cell given a fixed surface area
- Optimal tradeoff between studying math and studying chemistry given fixed time
- https://www.youtube.com/watch?v=czk4xgdhdY4

Global Maxima/minima

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- The global maximum of $f(x)$ over $[a, b]$ is always one of
  - A local maximum of $f(x)$, or
  - $f(a)$ or $f(b)$.
- The global maximum/minimum of $f(x)$ over the interval $[a, \infty)$ (if it exists!) is either a local maximum/minimum of $f(x)$, or is $f(a)$.

Warmup

Let $f(x) = -x^2 + 3x$.

Question: The global max/min of $f(x)$ on the interval $[3, 5]$ occur at $x =$

Question: The global max/min of $f(x)$ on the interval $[1, \infty)$ occur at $x =$

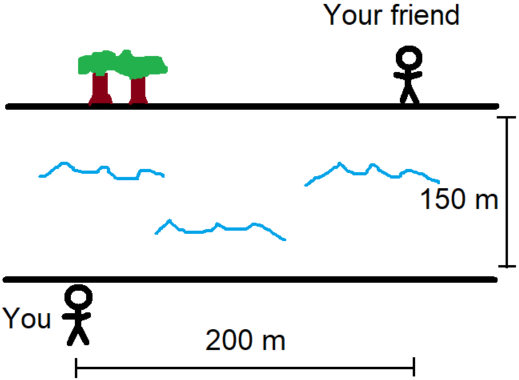
Warmup

Let $f(x) = -x^2 + 3x$.

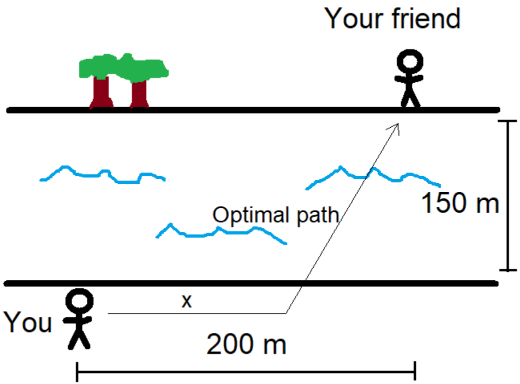
Question: The global max/min of $f(x)$ on the interval $[3, 5]$ occur at $x = 3$ (max) and $x = 5$ (min).

Question: The global max/min of $f(x)$ on the interval $[1, \infty)$ occur at $x = 1.5$ (max) and DNE (min).

Example: Crossing a river



- Swimming speed $=1$ meter per second
- Running speed $=4$ meters per second



Variables: x = distance run. Clearly doesn't make sense to have $x < 0$ or $x > 200$

Objective function: Time $= \frac{x}{4} + \frac{\sqrt{(200-x)^2+150^2}}{1}$

Bonus slide

We want to minimize the function $f(x) = \frac{x}{4} + \sqrt{(200-x)^2 + 150^2}$ over the interval $[0, 200]$.

- Find the CPs:

$$f'(x) = \frac{1}{4} - \frac{200-x}{\sqrt{(200-x)^2 + 150^2}} = 0$$

$$\frac{1}{4} = \frac{200-x}{\sqrt{(200-x)^2 + 150^2}}$$

$$\sqrt{(200-x)^2 + 150^2} = 4(200-x)$$

Bonus slide

$$(200 - x)^2 + 150^2 = 16(200 - x)^2$$

$$150^2 = 15(200 - x)^2$$

$$1500 = (200 - x)^2$$

$$x = 200 \pm 10\sqrt{15}$$

Only $\boxed{x = 200 - 10\sqrt{15}}$ lies in the interval $[0, 200]$.

Bonus slide

How do we know that $x = 200 - 10\sqrt{15}$ gives a minimum?

- Test the values $x = 0, 200, 200 - 10\sqrt{15}$.

$$f(0) = \frac{0}{4} + \sqrt{200^2 + 150^2} = 250$$

$$f(200) = \frac{200}{4} + \sqrt{150^2} = 200$$

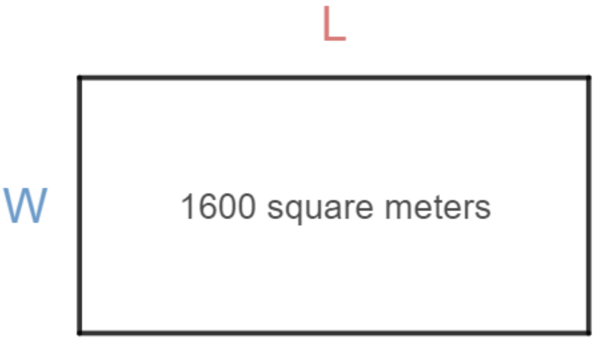
$$f(200 - 10\sqrt{15}) = \frac{200 - 10\sqrt{15}}{4} + \sqrt{1500 + 150^2}$$

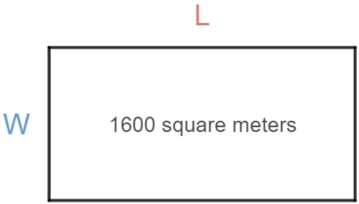
$$= 50 - \frac{5\sqrt{15}}{2} + 40\sqrt{15} = 50 + \frac{75\sqrt{15}}{2}$$

Since $\sqrt{15} < 4$, $50 + \frac{75\sqrt{15}}{2} < 50 + \frac{300}{2} = 200$.

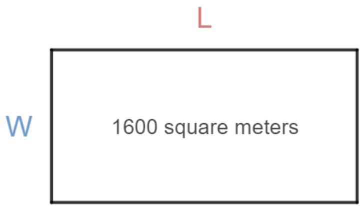
Example: Rectangle perimeter

Question: Given that a rectangle has a known area of 1600 square meters, but unknown length and width, what is the minimal possible perimeter?



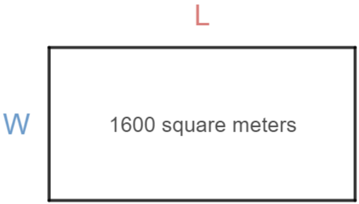


- **Variables:** L and W
- **Objective function:** $P = 2L + 2W$
- **Constraint:** $LW = 1600$



- **Variables:** L and W
- **Objective function:** $P = 2L + 2W$
- **Constraint:** $LW = 1600$

Thus, $W = \frac{1600}{L}$, and so $P = 2L + \frac{3200}{L}$.



- **Variables:** L and W
- **Objective function:** $P = 2L + 2W$
- **Constraint:** $LW = 1600$

Thus, $W = \frac{1600}{L}$, and so $P = 2L + \frac{3200}{L}$. P is minimized when $L = 40$.

Recap

- Global maxima and minima of a function over an interval
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